Determining the size of the proton

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Abstract

A measurement of the Lamb shift of 49,881.88(76) GHz in muonic hydrogen in conjunction with theoretical estimates of the proton structure effects was recently used to deduce an accurate but rather small radius of the proton. Such an important shift in the understanding of fundamental values needs reconfirmation. Using a different approach with electromagnetic form factors of the proton, we obtain a new expression for the transition energy, $\Delta = E_{2P_{3/2}}^{f=2} - E_{2S_{1/2}}^{f=1}$, in muonic hydrogen and deduce a proton radius, $r_p = 0.831$ fm.

I. INTRODUCTION

In a Quantum Field Theoretic (QFT) description of a muon-proton amplitude, the coupling of the photon to a point-like particle (vertex), $e\gamma_\mu$, has to be replaced by $e(F_1(q^2)\gamma_\mu + F_2(q^2)\sigma_{\mu\nu}q^\nu/2m_p)$ where q_μ is the four momentum carried by the photon, $q^2 = q_\mu q^\mu$ and $F_i(q^2)$ are the electromagnetic form-factors [1, 2] which encode the information on the structure of the proton. The inclusion of form factors introduces corrections to the binding energy in Coulomb bound systems such as the hydrogen atom. In fact a systematic inclusion of the proton structure effects introduces corrections to other terms such as the fine and hyperfine splittings in the hydrogen atom [3]. Given the recent high precision of experimental measurements, the corrections due to the proton size become relevant. One such precision measurement using pulsed laser spectroscopy was recently performed [4] to measure the Lamb shift in muonic hydrogen ($\mu^- p$). Using this accurate measurement of the energy difference (Lamb shift) between the $2S_{1/2}$ and $2P_{3/2}$ states, a precise value for the transition energy, $\Delta = E_{2P_{3/2}}^{f=2} - E_{2S_{1/2}}^{f=1} = 206.2949(32)$ meV, in muonic hydrogen was given. Comparing the measured value with a model calculation of this difference which included the proton structure effects, a new value of the proton radius, namely, $r_p = 0.84184(67)$ fm was published. It was further claimed that the new value implies that either the Rydberg constant has to be shifted by -110 kHz/c or the calculations of the QED effects in atomic hydrogen or muonic atom are insufficient. The uncertainties due to form factors have been discussed in [4]. Another uncertainty however seems to arise from the way in which one incorporates the finite size corrections (FSC) due to the structure of the proton. The FSC in [4] were based on methods given in [5, 6]. Here we implement the FSC based on the Breit equation [3, 7] and thus obtain a different form for the expression of the energy difference, namely, $\Delta(= E_{2P_{3/2}}^{f=2} - E_{2S_{1/2}}^{f=1}) = 209.16073 + 0.1139r_p - 4.3029r_p^2 + 0.02059r_p^3$ meV, as compared to, $\Delta^{Nature} = 209.9779(49) - 5.2262r_p^2 + 0.0347r_p^3$ meV, obtained in [4]. Based on this new expression and following the same procedure as in [4], we extract a new radius of the proton. This value of $r_p = 0.83112$ fm based on the FSC from the Breit equation is not far from that predicted in [4], namely $r_p = 0.84184(67)$. However, given the precision of the numbers under consideration, it is worth noting that the method used for proton structure corrections does introduce an uncertainty.

Before proceeding to the formalism used in the present work, we note certain points which

motivated the present work. The proton radius is one of the fundamental numbers in nuclear and particle physics and is obviously relevant for atomic physics too. The radius estimated in [4] is smaller than the value obtained by some other methods (see [4] for references and discussions on this issue). This could mean that some corrections to the energies in the hydrogen atom are missing. If a shift in the understanding of the proton radius occurs, it is necessary to reconfirm the result with an independent calculation. This is particularly so in view of the model dependence entering the extraction of the radius. The nuclear physics inputs (here, the structure of the proton) are never as precise as their electroweak counterparts. In the present work, we try to provide this reconfirmation using a different theoretical approach for the proton structure corrections. Though we do agree qualitatively on a smaller radius, we notice that the accuracy gets blurred by the approach used for finite size corrections.

II. BREIT POTENTIAL WITH FORM FACTORS

All $\vec{r}$ dependent potentials in Quantum Field Theory (QFT) are obtained by Fourier transforming an elastic scattering amplitude suitably expanded in $1/c^2$ (see [3] for several examples). The Breit equation [8, 9] follows the very same principle for elastic $e^-\mu^+$, e^+e^- (positronium), e^-p (hydrogen) and μ^-p (muonic hydrogen) amplitudes. The one-photon exchange amplitude between the proton and the muon leads then to the Coulomb potential plus the fine and hyperfine structure (hfs), the Darwin term and the retarded potentials [8, 9]. Here we use a modified Breit potential [3] for the μ^-p system which includes the electromagnetic form factors of the proton. We present a calculation based on this potential with form factors to evaluate the transition energy, Δ, and hence the proton radius, r_p, as in [4]. The Fourier transform of the momentum space Breit potential in [3] gives $V = -\alpha/r + \delta V(\hat{p}_\mu, \hat{p}_p, r)$, where $\delta V(\hat{p}_\mu, \hat{p}_p, r)$ contains Coulomb, Darwin, fine and hyperfine terms with form factors. The expectation values of these terms give the energy corrections to the various terms.

The standard Breit potential is normally written down either for point-like particles (with standard point-like vertices) or at zero momentum transfer at the vertices (i.e. $F_1(0)$ and $F_2(0)$), at least for the hadronic vertex. Therefore, extending it to include the full q^2 dependence of the proton form factors (which encodes the finite size corrections) is just a

straightforward and mild extension of the standard procedure. Thus one can obtain the Breit potential in momentum space for the e^-p or μ^-p system with the proton structure effects fully included and is given as (Eq. (21) in [3]):

$$
\hat{U}(\mathbf{p}_X, \mathbf{p}_p, \mathbf{q}) = 4\pi e^2 \Bigg[F_1^X F_1^p \Bigg(-\frac{1}{\mathbf{q}^2} + \frac{1}{8m_X^2 c^2} + \frac{1}{8m_p^2 c^2} + \frac{i\boldsymbol{\sigma}_p \cdot (\mathbf{q} \times \mathbf{p}_p)}{4m_p^2 c^2 \mathbf{q}^2} - \frac{i\boldsymbol{\sigma}_X \cdot (\mathbf{q} \times \mathbf{p}_X)}{4m_X^2 c^2 \mathbf{q}^2}
$$

$$
+ \frac{\mathbf{p}_X \cdot \mathbf{p}_p}{m_X m_p c^2 \mathbf{q}^2} - \frac{(\mathbf{p}_X \cdot \mathbf{q})(\mathbf{p}_p \cdot \mathbf{q})}{m_X m_p c^2 \mathbf{q}^4} - \frac{i\boldsymbol{\sigma}_p \cdot (\mathbf{q} \times \mathbf{p}_X)}{2m_X m_p c^2 \mathbf{q}^2} + \frac{i\boldsymbol{\sigma}_X \cdot (\mathbf{q} \times \mathbf{p}_p)}{2m_X m_p c^2 \mathbf{q}^2} + \frac{\boldsymbol{\sigma}_X \cdot \boldsymbol{\sigma}_p}{4m_X m_p c^2}
$$

$$
- \frac{(\boldsymbol{\sigma}_X \cdot \mathbf{q})(\boldsymbol{\sigma}_p \cdot \mathbf{q})}{4m_X m_p c^2 \mathbf{q}^2} \Bigg) + F_1^X F_2^p \Bigg(\frac{1}{4m_p^2 c^2} + \frac{i\boldsymbol{\sigma}_p \cdot (\mathbf{q} \times \mathbf{p}_p)}{2m_p^2 c^2 \mathbf{q}^2} - \frac{i\boldsymbol{\sigma}_p \cdot (\mathbf{q} \times \mathbf{p}_X)}{2m_X m_p c^2 \mathbf{q}^2} - \frac{(\boldsymbol{\sigma}_X \cdot \mathbf{q})(\boldsymbol{\sigma}_p \cdot \mathbf{q})}{4m_X m_p c^2 \mathbf{q}^2}
$$

$$
+ \frac{\boldsymbol{\sigma}_X \cdot \boldsymbol{\sigma}_p}{4m_X m_p c^2} \Bigg) + F_2^X F_1^p \Bigg(\frac{1}{4m_X^2 c^2} - \frac{i\boldsymbol{\sigma}_X \cdot (\mathbf{q} \times \mathbf{p}_X)}{2m_X^2 c^2 \mathbf{q}^2} + \frac{i\boldsymbol{\sigma}_X \cdot (\mathbf{q} \times \mathbf{p}_p)}{2m_X m_p c^2 \mathbf{q}^2} - \frac{(\boldsymbol{\sigma}_X \cdot \mathbf{q})(\boldsymbol{\sigma}_p \cdot \mathbf{q})}{4m_X m_p c^2 \mathbf{q}^2}
$$

$$
+ \frac{\boldsymbol{\sigma}_X \cdot \boldsymbol{\sigma}_p}{4m_X m_p c^2} \Bigg) + F_2^X F_2^p \Bigg(\frac{\boldsymbol{\sigma}_X \cdot \boldsymbol{\sigma}_p}{4m_X m_p c^2} - \frac{(\boldsymbol{\sigma}_X \cdot \mathbf{q})(\boldsymbol{\sigma}_p \cdot \mathbf{q})}{4m_X m_p c^2 \mathbf{q}^2} \Bigg) \Bigg], \tag{1}
$$

where, $X = e$ or μ. The Fourier transform of this potential results in a space dependent potential suitable for calculations of the corrections to the atomic energy levels via time-independent perturbation theory.

The two form factors, $F_1(q^2)$ and $F_2(q^2)$ of the proton are connected to the Sachs form factors, $G_E(q^2)$ and $G_M(q^2)$. The latter can be interpreted in the proton rest frame to be Fourier transforms of the charge and magnetization distributions in the proton. They can be approximated fairly well by a dipole form [1] (which is suitable for analytic calculations) as, $G_D(\mathbf{q}^2) = 1/(1 + \mathbf{q}^2/m^2)^2 \approx G_E^p(\mathbf{q}^2) \approx G_M^p(\mathbf{q}^2)/\mu_p$, where $(1 + \kappa_p) = \mu_p = 2.793$ is the proton's magnetic moment and m the dipole parameter. We use the standard non-relativistic approximation $q^2 \approx -\mathbf{q}^2$ to obtain the dipole form of $F_1^p(\mathbf{q}^2)$ and $F_2^p(\mathbf{q}^2)$ (see Eq.(30) in [3]). The parameter m is eventually related to the proton radius and taken as a free parameter to fix the radius of the proton (to be discussed below).

III. PROTON STRUCTURE CORRECTIONS TO ENERGIES

We follow a procedure similar to that in [4] to obtain the proton radius with the difference that the proton structure corrections are included via the Breit potential method. The Breit potential $V(r)$ as explained above consists of a series of terms [8] with the leading

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term being the one due to the Coulomb potential. Thus, the expectation values of $V(r) = V_C + V_D + V_{hfs} + V_{SO} + ...$ give rise to the various energy terms including the standard Coulomb, Darwin, fine structure, hyperfine structure etc. Below, we shall give the expression for some of the energies (using the Fourier transform of the momentum space potential with form factors [3]) which are relevant for the calculation of the radius. For details of the full potential in coordinate space, we refer the reader to [3].

A. Coulomb term

The first term, namely, the Coulomb potential, $\hat{V}_C^{FF}$, with form factors is given as $\hat{V}_C^{FF} = V_C + \Delta V_C$, i.e.,

$$\hat{V}_C^{FF} = -\frac{e^2}{r}\left[1 - \left(1 + \frac{\kappa_p}{(1-k^2)^2}\right)e^{-mr} - \left(1 + \frac{\kappa_p}{(1-k^2)}\right)\frac{m}{2}re^{-mr} + \frac{\kappa_p}{(1-k^2)^2}e^{-mkr}\right], \quad (2)$$

where $k = 2m_p/m$ with m_p being the mass of the proton and m the parameter entering the dipole form factor. Note that the above expression has been obtained under the condition that $2m_p \neq m$ and hence does not generate any singularities (which as such should not be expected too since the form factors are not singular). A small explanation regarding this and the other analytic expressions which follow is in order here. The potential in coordinate space is obtained by Fourier transforming the potential in momentum space. If we consider the Coulomb term (one with $F_1^X F_1^p(Q^2)/Q^2$ in Eq.(1), where we write $Q^2 = \mathbf{q}^2$), Fourier transform it and replace $F_1^p(Q^2)$ using the dipole form factor, we obtain

$$\hat{V}_C^{FF}(r) = -\frac{2e^2}{\pi r}\left[\int_0^\infty dQ \frac{\sin(Qr)m^4}{Q(m^2+Q^2)^2} + \kappa_p \int_0^\infty dQ \frac{Q\sin(Qr)m^4}{(m^2+Q^2)^2(4m_p^2+Q^2)}\right]. \quad (3)$$

We now use a partial fraction expansion to write the terms in the integrands. As an example, consider the second term in the square bracket of Eq.(3). We can see that

$$\frac{1}{(m^2+Q^2)^2(4m_p^2+Q^2)} = \frac{1}{(4m_p^2-m^2)^2(4m_p^2+Q^2)} - \frac{1}{(4m_p^2-m^2)^2(m^2+Q^2)}$$

$$+ \frac{1}{(4m_p^2-m^2)(m^2+Q^2)^2}$$

assuming of course that $4m_p^2 \neq m^2$. The integrals in (3) can be performed analytically and lead to Eq.(2) which contains terms of the type

$$\frac{1}{4m_p^2 - m^2} = -\frac{1}{m^2}\frac{1}{(1-k^2)}$$

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with $k = 2m_p/m$. Should it happen that $4m_p^2 = m^2$, then $1/[(m^2 + Q^2)^2(4m_p^2 + Q^2)] = [1/(m^2 + Q^2)^3]$ and the integral $\int_0^\infty dQ \, [Q \sin(Qr)/(m^2 + Q^2)^3]$ can be evaluated by differentiating twice the integral $I_1 = \int_0^\infty dQ \, [Q \sin(Qr)/(m^2 + Q^2)] = (\pi/2) \, e^{-mr}$ with respect to m. However, the case of $4m_p^2 = m^2$ seems unrealistic and is not considered in the present work.

The correction to the Coulomb energy due to form factors is then found as, $\Delta E_C = \langle nlm | \Delta V_C | nlm \rangle$, which with ΔV_C being only a function of r becomes, $\Delta E_C = \int_0^\infty \Delta V_C |R_{nl}(r)|^2 r^2 dr$ (with $R_{nl}(r)$ being the unperturbed hydrogen atom radial function as found in books). The correction to the Coulomb energy for any n, l is thus given by a lengthy expression involving hypergeometric functions (as explained in the appendix). Performing a series expansion of the hypergeometric function and truncating it at large orders of the fine structure constant α, we evaluate the correction to the energy in terms of the dipole parameter m. In order to rewrite the corrections in terms of the proton radius r_p, we perform an expansion of the type $G_E^p(q^2) = 1 - 2q^2/m^2 + ...$ of the Sachs dipole form factor and compare it with the standard $G_E^p(q^2) = 1 - <r^2> q^2/6 + ...$ One can then write $<r^2> = 12/m^2$ and convert the corrections for energies given in terms of m^2 to those in terms of $<r^2>$. For convenience, in what follows, we shall denote $<r^2>^{1/2} = r_p$, $<r^2> = r_p^2$, $<r^2>^{3/2} = r_p^3$ etc. Thus the corrections to the Coulomb terms of the $2S_{1/2}$ and $2P_{1/2}$ levels used in the present work can be rewritten in terms of the proton radius r_p as follows:

$$\Delta E_{Coul}^{2S_{1/2}} = \frac{m_r^3 \alpha^4}{24} (A_1 - A_2 + A_3) \, r_p^2 + \frac{m_r^4 \alpha^5}{12\sqrt{12}} (-2A_1 + 2\frac{A_2}{k} - 3A_3) r_p^3 \qquad (4)$$
$$+ \frac{21 m_r^5 \alpha^6}{4} (A_1 - \frac{A_2}{k^2} + 2A_3) \frac{r_p^4}{144} + \ldots,$$

with, $A_1 = 1 + [\kappa_p/(1 - k^2)^2]$, $A_2 = \kappa_p/[k^2(1 - k^2)^2]$, $A_3 = 1 + [\kappa_p/(1 - k^2)]$ and $k = 2m_p/m$.

$$\Delta E_{Coul}^{2P_{1/2}} = \frac{m_r^5 \alpha^6}{4} A \frac{r_p^4}{144} - \frac{m_r^6 \alpha^7}{2} B \frac{r_p^5}{(12)^{5/2}} + \ldots, \qquad (5)$$

where, $A = 3 + [\kappa_p/(1 - k^2)^2] + [2\kappa_p/(1 - k^2)] - [1/(k^4(1 - k^2)^2)]$ and $B = 7 + [2\kappa_p/(1 - k^2)^2] + [5\kappa_p/(1 - k^2)] - [2/(k^5(1 - k^2)^2)$, makes a negligible contribution to the finite size corrections. The Coulomb terms of the present work and those used in [4] will be compared in detail in section IV.

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B. Darwin term

Let us first look at the Darwin term in the standard Breit equation. The potential in coordinate space is given as:

$$\hat{V}_D^{\text{no } F_{1,2}(q^2)} = \frac{\pi e}{2m_X^2 c} \delta(\mathbf{r}) \left[1 + \frac{m_X^2}{m_p^2} \right]. \tag{6}$$

With the inclusion of the electromagnetic form factors,

$$\hat{V}_D^{F_{1,2}(q^2)} = \frac{e^2}{8m_X^2 c^2} \left[(1 + 2\kappa_X) G_1 + \frac{m_X^2}{m_p^2} G_2 \right], \tag{7}$$

which in the static limit, i.e. taking $F_{1,2}^p(q^2 = 0)$ reduces to

$$\hat{V}_D^{F_{1,2}(q^2=0)} = \frac{\pi e^2}{2m_X^2 c^2} \delta(\mathbf{r}) \left[1 + 2\kappa_X + \frac{m_X^2}{m_p^2} (1 + 2\kappa_p) \right]. \tag{8}$$

Here $X = e$ or μ depending on if we are considering the usual hydrogen atom or muonic hydrogen atom. In the above,

$$
\begin{aligned}
G_1 &= \left(1 + \frac{\kappa_p}{1-k} \right) \frac{m}{2} e^{-mr} + \frac{mk\kappa_p}{(1-k^2)^2} \frac{e^{-mr}}{r} - \frac{mk\kappa_p}{(1-k^2)^2} \frac{e^{-mkr}}{r}, \\
G_2 &= \left(1 + \kappa_p \left(\frac{1-2k}{1-k} \right) \right) \frac{m}{2} e^{-mr} - \frac{mk\kappa_p}{(1-k)^2} \frac{e^{-mr}}{r} + \frac{mk\kappa_p}{(1-k)^2} \frac{e^{-mkr}}{r}.
\end{aligned}
$$

Taking the expectation values of these potentials using first order time-independent perturbation theory, the corrections to the energies corresponding to the Darwin term can be found. For example, the Darwin term without form factors and for $l = 0$ is given as,

$$E_D^{\text{no } F_{1,2}(q^2)}(n, l = 0) = \frac{\alpha}{2m_X^2 c^2} \frac{1}{n^3 a_r^3} \left[1 + \frac{m_X^2}{m_p^2} \right], \tag{9}$$

and one with the inclusion of q^2 dependent proton form factors is

$$E_D^{F_{1,2}(q^2)}(n, l) = \frac{\alpha}{2m_X^2 c^2} \frac{1}{n^3 a_r^3} \left[(1 + 2\kappa_X) G_{D1}(n, l) + \frac{m_X^2}{m_p^2} G_{D2}(n, l) \right], \tag{10}$$

where $G_{D1}(n, l)$ and $G_{D2}(n, l)$ are lengthy expressions involving the hypergeometric functions, $_2F_1$. The Bohr radius, $a_r = 1/(m_r \alpha)$, where m_r is the reduced mass. If we restrict to form factors taken at $q^2 = 0$, then,

$$E_D^{F_{1,2}(q^2=0)}(n, l = 0) = \frac{\alpha}{2m_X^2 c^2} \frac{1}{n^3 a_r^3} \left[1 + 2\kappa_X + \frac{m_X^2}{m_p^2} (1 + 2\kappa_p) \right]. \tag{11}$$

In order to get a better insight into the expressions, we rather replace the hypergeometric function $_2F_1$ by its series expansion $_2F_1(a,b;c;z) = 1 + (ab/c)(z/1!) + [a(a+1)b(b+1)]/[c(c+1)](z^2/2!) + \ldots\ldots$ in the expressions for energies and truncate the series at large orders of the fine structure constant α. Substituting for the dipole parameter by $m^2 = 12/r_p^2$, the corrections to the energies corresponding to the Darwin terms with $n = 2$ and $l = 0, 1$ are given as

$$E_D^{F_{1,2}(q^2)}(2S_{1/2}) = \frac{m_r^3}{m_X^2}\frac{\alpha^4}{2^4}\left[\left((1+2\kappa_X) + \frac{m_X^2}{m_p^2}(1+2\kappa_p)\right) + \left((1+2\kappa_X)A_1 + \frac{m_X^2}{m_p^2}B_1\right)r_p(12)\right.$$
$$\left. + \left((1+2\kappa_X)A_2 + \frac{m_X^2}{m_p^2}B_2\right)r_p^2 + \left((1+2\kappa_X)A_3 + \frac{m_X^2}{m_p^2}B_3\right)r_p^3 + \ldots\right]$$

and

$$E_D^{F_{1,2}(q^2)}(2P_{1/2}) = \frac{m_r^3}{m_X^2}\frac{\alpha^4}{2^4}\left[\left((1+2\kappa_X)D_2 + \frac{m_X^2}{m_p^2}E_2\right)r_p^2 + \left((1+2\kappa_X)D_3 + \frac{m_X^2}{m_p^2}E_3\right)r_p^3 + \ldots\right].$$
$$(13)$$

The coefficients, A_1, B_1 etc in the above equations depend on κ_p and $k = 2m_p/m$. The correction for the $2P$ state turns out to be negligibly small and is not of much relevance for the present work. The coefficients which contribute to the $2S$ energy correction up to order r_p^2 are given as

$$A_1 = \frac{1}{a_r\sqrt{12}}\left[-6\left(1 + \frac{\kappa_p}{1-k^2}\right) - 4\frac{k^2\kappa_p}{(1-k^2)^2} + 4\frac{\kappa_p}{k(1-k^2)^2}\right]$$

$$A_2 = \frac{1}{48a_r^2}\left[84\left(1 + \frac{\kappa_p}{1-k^2}\right) + 42\frac{k^2\kappa_p}{(1-k^2)^2} - 42\frac{\kappa_p}{k^2(1-k^2)^2}\right]$$

$$B_1 = \frac{1}{a_r\sqrt{12}}\left[-6\left(1 + \kappa_p\frac{1-2k^2}{1-k^2}\right) + 4\frac{k^2\kappa_p}{(1-k^2)^2} - \frac{\kappa_p}{k(1-k^2)^2}\right]$$

$$B_2 = \frac{1}{48a_r^2}\left[84\left(1 + \kappa_p\frac{1-2k^2}{1-k^2}\right) - 42\frac{k^2\kappa_p}{(1-k^2)^2} + 42\frac{\kappa_p}{k^2(1-k^2)^2}\right].$$

Here, $a_r = 1/(m_r\alpha)$.

C. Fine structure

As mentioned in the beginning, the expectation values of the various terms in the potential in q-space, Eq. (1), give rise to the Coulomb, Darwin, retarded potential, fine and hyperfine structures in the hydrogen atom. The fifth and the ninth spin-orbit terms in this equation

are the ones corresponding to the fine structure. A Fourier transform of this potential leads to the following potential for the fine structure with form factors:

$$\hat{V}_{FS} = \frac{\alpha}{4m_X^2 c^2}\left[(1+2\kappa_X) + \frac{2m_X}{m_p}(1+\kappa_X)\right]\left(\frac{1}{r^3} + \frac{G_{FS}}{r^3}\right)\mathbf{L}.\mathbf{S}_X, \qquad (14)$$

where,

$$G_{FS} = -\left(1 + \frac{\kappa_p}{(1-k)^2}\right)e^{-mr}(1+mr) - \left(1 + \frac{\kappa_p}{(1-k)}\right)\frac{m}{2}r^2 e^{-mr}$$
$$+ \frac{\kappa_p}{(1-k)^2}e^{-mkr}(1+mkr).$$

Using first order time-independent perturbation theory for evaluating the expectation value of the above potential, the fine structure energy term with the effect of form factors is given as,

$$E_{FS}(n,l,j) = \frac{\alpha}{4m_X^2 c^2}\left[(1+2\kappa_X) + \frac{2m_X}{m_p}(1+\kappa_X)\right]\left(j(j+1) - l(l+1) - s_X(s_X+1)\right)$$
$$\left(\frac{1}{l(l+1)(l+\frac{1}{2})n^3 a_r^3} + G_{FS}(n,l)\right), \qquad (15)$$

where, $G_{FS}(n,l)$ is given by a lengthy expression which contains the Gamma functions and hypergeometric functions. The electron (or muon) total angular momentum $j = l + s_X$, where l is the orbital angular momentum and s_X the spin (with $X = e$ or μ). The fine structure levels relevant for the present work are the $2P_{3/2}$ and $2P_{1/2}$ levels which appear in the difference, $\Delta E_{FS}^{2P_{3/2}} = E_{FS}^{2P_{3/2}} - E_{FS}^{2P_{1/2}}$. The proton structure corrections to the $2P$ levels are always much smaller than those to the $2S$ levels, however, for completeness we give these expressions below.

$$E_{FS}^{2P_{1/2}} = -\frac{m_r^3 \alpha^4}{48 m_X^2 c^2}\left(1 + 2\kappa_X + 2\frac{m_X}{m_p}(1+\kappa_X)\right)(1 + A_{FS}r_p^2) + \ldots, \qquad (16)$$

where,

$$A_{FS} = \frac{1}{4a_r^2}\left[-\left(1 + \frac{\kappa_p}{(1-k^2)^2}\right) - \left(1 + \frac{\kappa_p}{1-k^2}\right) + \frac{\kappa_p}{k^2(1-k^2)^2}\right],$$

with, $a_r = 1/(m_r\alpha)$ as before. The energy of the $2P_{3/2}$ level with form factor corrections is given by the relation $E_{FS}^{2P_{3/2}} = -(1/2)E_{FS}^{2P_{1/2}}$.

D. Hyperfine structure

The hyperfine potential and the corresponding correction to the energy due to form factors in the case of muonic hydrogen has been discussed in detail in Ref. [3]. Here we give

a brief description of this calculation for completeness. The $2S$ and $2P$ hyperfine energies relevant for this work are evaluated using the hyperfine potential,

$$\hat{V}_{hfs}(r) = \frac{\alpha\mu_p}{4r^3 m_X m_p c^2}\left[\mu_X\left\{3(\boldsymbol{\sigma}_X\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_p\cdot\hat{\mathbf{r}})f_1(r) - \boldsymbol{\sigma}_X\cdot\boldsymbol{\sigma}_p f_2(r)\right\} + 2\mathbf{L}\cdot\boldsymbol{\sigma}_p f_3(r)\right], \quad (17)$$

where, $\mu_X = 1 + \kappa_X$,

$$f_1(r) = 1 - e^{-mr}(1 + mr) - \frac{m^2 r^2}{6}e^{-mr}(3 + mr),$$

$$f_2(r) = f_1(r) - (m^3 r^3/3)e^{-mr} \text{ and,}$$

$$f_3(r) = 1 - e^{-mr}(1 + mr) - \frac{m^2 r^2}{2}e^{-mr}.$$

The calculation of energies for states with $l = 0$ and $l \neq 0$ is done separately [3]. We will see in section IV that the hyperfine levels of relevance are $E_{hfs}^{2S_{1/2}^{f=1}}$, $E_{hfs}^{2S_{1/2}^{f=0}}$, $E_{hfs}^{2P_{3/2}^{f=2}}$ and $E_{hfs}^{2P_{3/2}^{f=1}}$. The complete expressions for any n and l can be found in [3]. For the $2S_{1/2}$ case,

$$E_{hfs}^{2S_{1/2}^{f=1}} = \frac{\alpha^4 m_r^3}{m_\mu m_p}\frac{(1+\kappa_\mu)(1+\kappa_p)}{12}\left(1 - \frac{6}{\sqrt{12}a_r}r_p + \frac{21}{12a_r^2}r_p^2 - \frac{55}{12^{3/2}a_r^3}r_p^3 + ...\right), \quad (18)$$

with $a_r = 1/(m_r\alpha)$. $E_{hfs}^{2S_{1/2}^{f=0}} = -3E_{hfs}^{2S_{1/2}^{f=1}}$ and $(1/4)\Delta E_{hfs}^{2S} = E_{hfs}^{2S_{1/2}^{f=1}}$. Note the presence of the term linear in r_p. The sum of this correction and one arising from the Darwin term will contribute a small term linear in r_p to the final expression for the transition energy, Δ. Such a term linear in r_p does not exist in the calculation of Pohl et al. [4]. The energies of the P levels are given as,

$$E_{hfs}^{2P_{3/2}^{f=1}} = \left(\frac{C_1}{6} - \frac{5C_2}{6} - \frac{5C_3}{6}\right) + \left(\frac{C_1}{3} + 5C_2 - \frac{5C_4}{6}\right)\frac{r_p^2}{12a^2} \quad (19)$$
$$+ \left(-\frac{10C_1}{3} - \frac{50C_2}{3} - \frac{5C_5}{6}\right)\frac{r_p^3}{12^{3/2}a^3} + \cdots,$$

where

$$C_1 = \frac{m_r^3\alpha^4}{m_\mu m_p}\frac{(1+\kappa_\mu)(1+\kappa_p)}{24}, \quad C_2 = \frac{m_r^3\alpha^4}{m_\mu m_p}\frac{(1+\kappa_p)}{24},$$

$$C_3 = \frac{m_r^3\alpha^4}{24m_\mu m_p}\left(\frac{m_\mu}{2m_p}\right)(1 + 2\kappa_p), \quad C_4 = \frac{m_r^3\alpha^4}{24m_\mu m_p}\left(\frac{m_\mu}{2m_p}\right)\left(-6(1 + 2\kappa_p) - 3\frac{\kappa_p}{k^2}\right)$$

$$C_5 = \frac{m_r^3\alpha^4}{24m_\mu m_p}\left(\frac{m_\mu}{2m_p}\right)\left(20(1 + 2\kappa_p) - \frac{8\kappa_p}{(1 - k^2)^2} + \frac{8\kappa_p}{k^3(1 - k^2)^2} - \frac{12\kappa_p}{1 - k^2}\right).$$

$$E_{hfs}^{2P_{3/2}^{f=2}} = \left(-\frac{C_1}{10} + \frac{C_2}{2} + \frac{C_3}{2}\right) + \left(3C_1 - 3C_2 + \frac{C_4}{2}\right)\frac{r_p^2}{12a^2} \quad (20)$$
$$+ \left(-14C_1 + 10C_2 + \frac{C_5}{2}\right)\frac{r_p^3}{12^{3/2}a^3} + \cdots$$

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with C_1, C_2, C_3 and C_4 the same as in the $2P_{3/2}^{f=1}$ case.

The contribution of the P levels to the finite size corrections are once again small.

IV. RADIUS OF THE PROTON

In the next step, we evaluate the radius by calculating the energy corrections to expressions dependent on the proton radius r_p. Following a similar procedure as in [4] conceptually, we perform a fit using the experimental value of the transition energy, $\Delta = E_{2P_{3/2}}^{f=2} - E_{2S_{1/2}}^{f=1}$ and the theoretical expressions for the corresponding energies (with finite size corrections) to determine the radius, r_p. In order to evaluate the energy difference, $\Delta = E_{2P_{3/2}}^{f=2} - E_{2S_{1/2}}^{f=1}$, we need to find as in [4],

$$E_{2S_{1/2}^{f=1}} = \frac{1}{4}\Delta E_{hfs}^{2S}, \tag{21}$$

$$E_{2P_{3/2}^{f=2}} = \Delta E_{LS} + \Delta E_{FS}^{2P_{3/2}} + \frac{3}{8}\Delta E_{hfs}^{2P_{3/2}},$$

$$\Delta E_{LS} = E_{2P_{1/2}} - E_{2S_{1/2}}.$$

Here $f = j + s_p$ is the total angular momentum of the muon proton system with the muon total angular momentum, $j = l + s_\mu$. The fine and hyperfine splittings respectively are,

$$\Delta E_{FS}^{2P_{3/2}} = E_{FS}^{2P_{3/2}} - E_{FS}^{2P_{1/2}}, \tag{22}$$

$$\Delta E_{hfs}^{2P_{3/2}} = E_{hfs}^{2P_{3/2}^{f=2}} - E_{hfs}^{2P_{3/2}^{f=1}}.$$

ΔE_{hfs}^{2S} is the hyperfine splitting of the $2S$ levels, i.e., $\Delta E_{hfs}^{2S} = E_{hfs}^{2S_{1/2}^{f=1}} - E_{hfs}^{2S_{1/2}^{f=0}}$. Such a fit in our case leads to the proton radius $r_p = 0.83112$ fm. The correction to the Coulomb term evaluated from Eq. (4) gives $\Delta E_{Coul}^{2S_{1/2}} = 4.30248r_p^2 - 0.020585r_p^3$ meV, which forms the major contribution to the r_p^2 and r_p^3 terms to be discussed below.

A. Coefficients of the r_p^2 and r_p^3 terms

The value -5.2262 r_p^2 + 0.0347 r_p^3 meV used in [4] for $-\Delta E_{Coul}^{2S_{1/2}}$ is based on the following equation taken from [10] which is based on Friar's formalism [6]:

$$\Delta E^{Borie} = \frac{2\alpha Z}{3}\left(\frac{\alpha Z m_r}{n}\right)^3\left[< r^2 > - \frac{\alpha Z m_r}{2} < r^3 >_{(2)} + ...\right]. \tag{23}$$

In this formalism, the correction to the $1/r$ Coulomb potential takes the form

$$\Delta V_c(r) = -Z\alpha \int d^3s \, \rho(s) \left(\frac{1}{|\mathbf{r} - \mathbf{s}|} - \frac{1}{r} \right) \tag{24}$$

where $\rho(s)$ is the charge distribution in the proton. Using perturbation theory, the correction to the energy level is evaluated in [6] as (see Eq. (43a) in [6]),

$$\Delta E \simeq \frac{2\pi Z\alpha}{3} |\phi_n(0)|^2 \left(<r^2> - \frac{Z\alpha\mu}{2} <r^3>_{(2)} + \dots \right). \tag{25}$$

In contrast to our formalism where we consider the full r dependent wave function and perform the integration, Ref. [6] considers the wave function only at $r = 0$, as a result of which one is left with the integral $<r^2> = \int d^3r \, \rho(r) \, r^2$ in the energy correction. First we discuss the $<r^2>$ dependent term and then go over to the term with $<r^3>_{(2)}$.

The energy corrections and hence the coefficients of the r_p^2 (writing $<r^2> = r_p^2$ as in [4]) term as calculated from Friar's formalism will clearly differ from the present work due to the difference in Eqs (24) and (2) for the two potentials. The finite size correction to the to Coulomb energy of the present work has been evaluated using the potential which is a Fourier transform of the first term in Eq.(1), namely,

$$U_{Coul}(\mathbf{q}) = 4\pi e^2 \, F_1^X F_1^p \left(-\frac{1}{\mathbf{q}^2} \right).$$

If instead of taking just the Coulomb potential in momentum space, we decide to club one of the Darwin terms with it, we find

$$U_{Coul}^{newdef}(\mathbf{q}) = 4\pi e^2 \left[F_1^X F_1^p \left(-\frac{1}{\mathbf{q}^2} \right) + F_1^X F_2^p \left(\frac{1}{4m_p^2 c^2} \right) \right] \tag{26}$$

$$= -4\pi\alpha \left[\frac{G_E^p(\mathbf{q}^2)}{\mathbf{q}^2} \right].$$

A Fourier transform of this potential with a dipole form factor leads to the following potential in coordinate space:

$$V_C^{newdef} = -\frac{\alpha}{r} \left[1 - e^{-mr} \left(1 + \frac{mr}{2} \right) \right]. \tag{27}$$

The energy correction evaluated for the $2S$ state (i.e. n=2, l=0) using $\Delta V_C = V_C^{newdef} - (-\alpha/r)$ in $\Delta E_C = \int_0^\infty \Delta V_C |R_{nl}(r)|^2 r^2 dr$ is then

$$\Delta E_{Coul}^{newdef} = \left(\frac{1}{2a_r} \right)^3 \frac{\alpha}{m^2} \left[\left(\frac{ma_r}{1 + ma_r} \right)^2 (1 + 3F(-2, 2; 3; 2/(1 + ma_r))) + \tag{28}$$

$$\left(\frac{ma_r}{1 + ma_r} \right)^3 (1 + 3F(-2, 3; 3; 2/(1 + ma_r))) \right]. \tag{29}$$

Expressing the hypergeometric function as a series expansion, truncating it at high orders in α and substituting $r_p^2 = 12/m^2$ leads to

$$\Delta E_{Coul}^{newdef} = \frac{\alpha^4 m_r^3}{12} \left(r_p^2 - \frac{5\alpha m_r}{\sqrt{12}} r_p^3 + \dots \right). \tag{30}$$

The first term in Eq.(30) is exactly equal to that in (23) for the $2S$ state. This means that in principle, taking one of the Darwin terms together with the Coulomb one in the Breit potential leads to an exact agreement with the coefficient of the r_p^2 term used in [4]. However, it also means that there are cancellations due to the other Darwin terms which eventually leaves us with a smaller coefficient of the r_p^2 term than that used in [4].

The second term in Eq.(23) is the third moment of the convoluted proton charge density and is defined as,

$$< r^3 >_{(2)} = \int d^3r \, r^3 \, \rho_{(2)}(r) \tag{31}$$

where the convoluted charge density is given by

$$\rho_{(2)} = \int d^3z \, \rho_{ch}(|\mathbf{z} - \mathbf{r}|) \, \rho_{ch}(z). \tag{32}$$

Inserting the Fourier transform of the Sachs electric form factor $G_E(Q^2)$ for ρ_{ch}, one finds that

$$< r^3 >_{(2)} = \frac{48}{\pi} \int_0^\infty \frac{dQ}{Q^4} \left(G_E^2(Q^2) - 1 + \frac{Q^2}{3} < r^2 > \right). \tag{33}$$

This third moment comes about due to the use of the smeared Coulomb potential (Eq. (24)) along with the perturbative expansion of the hydrogen wave functions. Using a dipole form factor G_E, one can show that $< r^3 >_{(2)} = 35\sqrt{3} < r^2 >^{3/2}$. This substitution in Eq. (23) indeed leads to the factor $0.0347 \, r_p^3$ in [4]. Such a correction of order α^5 which is proportional to the proton form factor squared is not included in the present work. Including a term with the proton form factor squared in the Breit potential would correspond to a two photon exchange diagram which would obviously lead to a correction one order higher in α and hence would be quite small.

B. Darwin contribution to the Lamb shift and the fine and hyperfine corrections

The main contribution to the Lamb shift ΔE_{LS} comes from QED, namely, the Uehling potential. In the Nature paper [4] using other methods [5, 6] for including the proton

structure corrections, the authors found

$$\Delta E_{LS}^{Nature} = 206.0573(45) - 5.2262 r_p^2 + 0.0347 r_p^3 \, \text{meV}, \tag{34}$$

where the first term includes the relativistic one loop vacuum polarization for a point nucleus plus other QED corrections and the second and third terms are due to finite size corrections (FSC). To evaluate ΔE_{LS} with FSC with the Breit potential approach, apart from $\Delta E_{Coul}^{2S_{1/2}} = 4.30248 r_p^2 - 0.020585 r_p^3 \, \text{meV}$, we need the corrections $\Delta E_{Coul}^{2P_{1/2}}$ and the $2S$ and $2P$ corrections to the Darwin terms too. The structure corrections to the Darwin and Coulomb terms of the $2P$ levels are negligibly small. If we use Eqs (9) and (11) for the $2S_{1/2}$ level, we get, $E_D^{no\,F_{1,2}(q^2)} = 13.768591 \, \text{meV}$ and $E_D^{F_{1,2}(q^2=0)} = 14.4185121 \, \text{meV}$. Starting with Eq.(10) and following the same procedure of expanding the hypergeometric functions etc, we find for the $2S_{1/2}$ level,

$$E_D^{F_{1,2}(q^2)} = 14.418512 - 0.0793 \, r_p + 0.0002613 \, r_p^2 - 6.6 \times 10^{-6} \, r_p^3 + \dots \text{meV}. \tag{35}$$

Note that the first term in the above equation is nothing but the Darwin term with form factors taken at $q^2 = 0$. Since the calculation of the Lamb shift as used in [4] has been taken from a relativistic calculation, in principle we do not need to take into account the Darwin term. However, that calculation was done without including proton form factors and hence we must include the "correction" to the Darwin term due to form factors. Thus, we subtract $E_D^{no\,F_{1,2}(q^2)} = 13.768591 \, \text{meV}$ from Eq. (35) and use

$$\Delta E_{Darwin}^{2S_{1/2}} = 0.64992 - 0.0793 r_p + 0.0002613 r_p^2 - 6.6 \times 10^{-7} \, r_p^3 \, \text{meV}, \tag{36}$$

which is the correction to the Darwin term due to the proton form factors. Besides these, we also have, $\Delta E_{Coul}^{2P_{1/2}} = 3 \times 10^{-6} r_p^4 - 1.37 \times 10^{-8} r_p^5 \, \text{meV}$, $\Delta E_{Darwin}^{2P_{1/2}} = 9.08 \times 10^{-8} r_p^2 - 4.4 \times 10^{-10} r_p^3 \, \text{meV}$ which are however extremely small corrections.

To evaluate ΔE_{LS}, we start as in [4] with the value 206.0573 which is the sum of 24 terms from the table given in the supplementary material of [4]. To this we add the FSC mentioned above for $2S_{1/2}$ and $2P_{1/2}$ levels and find $\Delta E_{LS} = E_{2P_{1/2}} - E_{2S_{1/2}}$ to be,

$$\Delta E_{LS} = 205.40738 + 0.0793 r_p - 4.30274 r_p^2 + 0.020585 r_p^3 \, \text{meV}, \tag{37}$$

which should be compared with ΔE_{LS}^{Nature} mentioned in (34) above. Our result for ΔE_{LS} with FSC includes an additional term in r_p which is not present in ΔE_{LS}^{Nature}. The coefficients

of the r_p^2 and r_p^3 terms are however quite similar. The corrections to the P levels are really tiny as expected.

Finally, the proton size corrections to the hyperfine and fine structure energy levels are evaluated as explained in the previous section. The fine and hyperfine structure terms in (21) and (22) are expressed in terms of the proton radius as follows:

$$\Delta E_{FS}^{2P_{3/2}} = 8.34678 - 4.26 \times 10^{-5} r_p^2 + 1.36 \times 10^{-7} r_p^3 \, \text{meV} \tag{38}$$

$$\Delta E_{hfs}^{2P_{3/2}} = 3.3912 - 1.787 \times 10^{-5} r_p^2 + 5.45 \times 10^{-8} r_p^3 \, \text{meV},$$

$$(1/4)\Delta E_{hfs}^{2S} = 5.708 - 0.0347 r_p + 0.0001 r_p^2 - 3.27 \times 10^{-7} r_p^3 \, \text{meV}.$$

One can see that the FSC for fine structure are very small. The first term 8.34678 meV is exactly the same as the sum of first two terms given in Table I in the second reference of [11].

In the evaluation of the proton radius in Pohl et al. [4], the values of the hyperfine splittings were taken from [11], where the FSC for the $2S$ level were evaluated using the Zemach method and those for the $2P$ case were not taken into account. Their FSC (taken from Table II of the first reference in [11]) of order α^5 and α^6 sum to -0.1535 meV. This correction is obtained using a Zemach radius of $R_Z = 1.022$ fm and leads to $\Delta E_{hfs}^{2S} = 22.8148$ meV. We note that using such a value of ΔE_{hfs}^{2S} means that the information about the radius of the proton (through the form factors or Zemach radius $R_Z = 1.022$ fm) has already been included in the hyperfine energy which is later used to extract the radius of the proton. However, if we wish to remain within the spirit of the formalism used for ΔE_{LS}, then we cannot take an input energy which already assumes a certain radius of the proton. Hence, instead of using just a number for ΔE_{hfs}^{2S} (as done in [4]) which already includes the FSC, we use the expressions given above in terms of r_p.

Note that the above equations include the proton structure corrections *only*. Adding QED and other corrections to these levels as in Refs. [14,15] in [4] and following a procedure similar to that in [4] (thus differing only in the inclusion of the FSC), we obtain

$$\Delta(= E_{2P_{3/2}}^{f=2} - E_{2S_{1/2}}^{f=1}) = 209.16073 + 0.11388 r_p - 4.3029 r_p^2 + 0.020585 r_p^3 \, \text{meV}, \tag{39}$$

as compared to,

$$\Delta^{Nature} = 209.9779(49) - 5.2262 r_p^2 + 0.0347 r_p^3 \, \text{meV}, \tag{40}$$

obtained in [4].

Eq. (39) contains a term linear in r_p which is not present in the expression (40) of Pohl et al [4]. This term arises partly due to the proton structure correction to the Darwin term and partly due to the $2S$ hyperfine energy level. For the $2S$ hyperfine level, this is a correction of order α^5 as can be seen from Eq. (18). In [4], such an order α^5 correction of -0.1518 meV (see Table II in the first reference in [11] which was used in [4]) has been included directly as a number while taking into account the energy of the $2S$ hyperfine level (see also the discussion in the paragraph below Eq.(38)). The remaining part of the r_p dependent term in our formalism arises from the form factor correction to the Darwin term which contributes to the expression of the Lamb shift in (37). In [4] the main contribution of 205.0282 meV to the Lamb shift arises from the relativistic one loop vacuum polarization which has been evaluated in [10] for a point nucleus. Hence, though the relativistic effects (represented by the Darwin term in our formalism) are taken into account, the finite size corrections to this term are missing in [4].

Taking the central value of the measured $\Delta = 206.2949(32)$ meV and replacing in the left hand side of Eq.(39) leads to $r_p = 0.83112$ fm. If we compare this radius with $r_p = 0.84184(67)$ fm (obtained from (40)) in [4] we see that an additional uncertainty arises due to the difference in the approaches used for including proton finite size effects. A detailed comparison of the present approach with those used in [4] can be found in section IV of Ref. [3].

To conclude, we can say that it is gratifying to see that in spite of the differences in the approaches for calculating the FSC, we obtain $r_p = 0.83112$ fm which is not too far from the value of $r_p = 0.84184(67)$ fm found in [4]. It is however important to note that there exist different approaches [3, 5, 6] for evaluating the finite size effects in literature which can lead to different results (and an additional uncertainty in the extracted radius). One should be cautious not to overestimate the accuracy in calculating the radius of the proton.

Appendix: Evaluation of energies

The energies corresponding to the Coulomb, Darwin, fine structure and hyperfine structure terms in the Breit equation are evaluated by taking the expectation value of the corresponding potential using first order time-independent perturbation theory. In general, for

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an operator $\hat{\mathbf{A}}$, the expectation value is,

$$\langle \hat{\mathbf{A}} \rangle = \int r^2 \, dr \, d\theta \, d\phi \, \Psi^*_{nlm_l}(r, \theta, \phi) \hat{\mathbf{A}} \Psi_{nlm_l}(r, \theta, \phi), \tag{A.1}$$

where,

$$\Psi(r, \theta, \phi) = R_{nl}(r) Y_l^{m_l}(\theta, \phi), \tag{A.2}$$

with

$$R_{nl}(r) = \left[\left(\frac{2}{na} \right)^3 \frac{(n-l-1)!}{2n(n+l)!} \right]^{1/2} e^{-r/na} \left(\frac{2r}{na} \right)^l L_{n-l-1}^{2l+1}(2r/na) \tag{A.3}$$

being the radial functions. For operators which do not depend on angles, the expectation value reduces to calculating

$$\langle \hat{\mathbf{A}} \rangle = \left(\frac{2}{na} \right)^{2l+3} \frac{1}{2n2^{2(n-l-1)}} \sum_{j=0}^{n-l-1} \binom{2(n-l-j-1)}{n-l-j-1} \frac{(2j)!}{j!\Gamma(2l+j+2)}$$
$$\times \int_0^\infty dr \mathbf{A} e^{-2r/na} r^{2l+2} L_{2j}^{2(2l+1)}(4r/na). \tag{A.4}$$

The above integral can be evaluated analytically for the potentials considered in the present work. This amounts to evaluating the expectation values of terms of the type e^{-mr}, re^{-mr}, e^{-mr}/r etc. For example,

$$\langle e^{-mr} \rangle = \left(\frac{2}{na} \right)^{2l+3} \frac{1}{2n2^{2(n-l-1)}} \sum_{j=0}^{n-l-1} \binom{2(n-l-j-1)}{n-l-j-1} \frac{\Gamma(4l+2j+3)}{j!\Gamma(2l+j+2)}$$
$$\frac{\Gamma(2l+3)}{\Gamma(4l+3)} \left(\frac{na}{2+mna} \right)^{2l+3} F\left(-2j, 2l+3; 4l+3; \frac{4}{2+mna} \right) \tag{A.5}$$

$$\left\langle \frac{e^{-mr}}{r} \right\rangle = \left(\frac{2}{na} \right)^{2l+3} \frac{1}{2n2^{2(n-l-1)}} \sum_{j=0}^{n-l-1} \binom{2(n-l-j-1)}{n-l-j-1} \frac{\Gamma(4l+2j+3)}{j!\Gamma(2l+j+2)}$$
$$\frac{\Gamma(2l+2)}{\Gamma(4l+3)} \left(\frac{na}{2+mna} \right)^{2l+2} F\left(-2j, 2l+2; 4l+3; \frac{4}{2+mna} \right) \tag{A.6}$$

$$\left\langle \frac{e^{-mr}}{r^2} \right\rangle = \left(\frac{2}{na} \right)^{2l+3} \frac{1}{2n2^{2(n-l-1)}} \sum_{j=0}^{n-l-1} \binom{2(n-l-j-1)}{n-l-j-1} \frac{\Gamma(4l+2j+3)}{j!\Gamma(2l+j+2)}$$
$$\frac{\Gamma(2l+1)}{\Gamma(4l+3)} \left(\frac{na}{2+mna} \right)^{2l+1} F\left(-2j, 2l+1; 4l+3; \frac{4}{2+mna} \right) \tag{A.7}$$

$$\left\langle \frac{e^{-mr}}{r^3} \right\rangle = \left(\frac{2}{na} \right)^{2l+3} \frac{1}{2n2^{2(n-l-1)}} \sum_{j=0}^{n-l-1} \binom{2(n-l-j-1)}{n-l-j-1} \frac{\Gamma(4l+2j+3)}{j!\Gamma(2l+j+2)}$$
$$\frac{\Gamma(2l)}{\Gamma(4l+3)} \left(\frac{na}{2+mna} \right)^{2l} F\left(-2j, 2l; 4l+3; \frac{4}{2+mna} \right), \tag{A.8}$$

with the expressions being dependent on the Gamma functions and hypergeometric functions. The latter can be expressed as,

$$
\begin{aligned}
F(a, b; c; z) &= 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \dots \\
&= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.
\end{aligned} \tag{A.9}
$$

The expectation value of the Coulomb potential with form factors in Eq.(2) for example leads to the Coulomb potential plus the finite size correction to the Coulomb energy for any n, l which is given by

$$
\begin{aligned}
\Delta E_{Coul}(n, l) = \alpha \left(\frac{2}{na_r}\right)^{2l+3} \frac{1}{2n2^{2(n-l-1)}\Gamma(4l+3)} \sum_{j=0}^{n-l-1} \binom{2(n-l-j-1)}{n-l-j-1} \frac{\Gamma(4l+2j+3)}{j!\Gamma(2l+j+2)} \\
\times \left[\left(1 + \frac{\kappa_p}{(1-k^2)^2}\right)\Gamma(2l+2)\left(\frac{na_r}{2+mna_r}\right)^{2l+2} F\left(-2j, 2l+2; 4l+3; \frac{4}{2+mna_r}\right)\right. \\
- \frac{\kappa_p}{(1-k^2)^2}\Gamma(2l+2)\left(\frac{na_r}{2+mkna_r}\right)^{2l+2} F\left(-2j, 2l+2; 4l+3; \frac{4}{2+mkna_r}\right) \\
\left. + \frac{m}{2}\left(1 + \frac{\kappa_p}{1-k^2}\right)\Gamma(2l+3)\left(\frac{na_r}{2+mna_r}\right)^{2l+3} F\left(-2j, 2l+3; 4l+3; \frac{4}{2+mna_r}\right)\right]. \tag{A.10}
\end{aligned}
$$

The energy corresponding to the Darwin term for example can be calculated by taking the expectation value of the potential in (7) and leads to Eq.(10), namely,

$$
E_D^{F_{1,2}(q^2)}(n, l) = \frac{\alpha}{2m_X^2 c^2}\frac{1}{n^3 a_r^3}\left[(1 + 2\kappa_X)G_{D1}(n, l) + \frac{m_X^2}{m_p^2}G_{D2}(n, l)\right], \tag{A.11}
$$

where $G_{D1}(n, l)$ and $G_{D2}(n, l)$ are given as,

$$
\begin{aligned}
G_{D1}(n, l) = \left(\frac{2}{na_r}\right)^{2l} \frac{1}{n2^{2(n-l-1)}} \sum_{j=0}^{n-l-1} \binom{2(n-l-j-1)}{n-l-j-1} \frac{\Gamma(4l+2j+3)}{j!\,\Gamma(2l+j+2)\Gamma(4l+3)} \\
\times \left[\left(1 + \frac{\kappa_p}{1-k}\right)\frac{m^3}{2}\Gamma(2l+3)\left(\frac{na_r}{2+mna_r}\right)^{2l+3} F\left(-2j, 2l+3; 4l+3; \frac{4}{2+mna_r}\right)\right. \\
+ \frac{m^2k^2\kappa_p}{(1-k^2)^2}\Gamma(2l+2)\left(\frac{na_r}{2+mna_r}\right)^{2l+2} F\left(-2j, 2l+2; 4l+3; \frac{4}{2+mna_r}\right) \\
\left. - \frac{m^2k^2\kappa_p}{(1-k^2)^2}\Gamma(2l+2)\left(\frac{na_r}{2+mkna_r}\right)^{2l+2} F\left(-2j, 2l+2; 4l+3; \frac{4}{2+mkna_r}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
G_{D2}(n, l) = \left(\frac{2}{na_r}\right)^{2l} \frac{1}{n2^{2(n-l-1)}} \sum_{j=0}^{n-l-1} \binom{2(n-l-j-1)}{n-l-j-1} \frac{\Gamma(4l+2j+3)}{j!\,\Gamma(2l+j+2)\Gamma(4l+3)} \\
\times \left[\left(1 + \kappa_p\left(\frac{1-2k^2}{1-k^2}\right)\right)\frac{m^3}{2}\Gamma(2l+3)\left(\frac{na_r}{2+mna_r}\right)^{2l+3} F\left(-2j, 2l+3; 4l+3; \frac{4}{2+mna_r}\right)\right.
\end{aligned}
$$

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$$-\frac{m^2 k^2 \kappa_p}{(1-k^2)^2}\Gamma(2l+2)\left(\frac{na_r}{2+mna_r}\right)^{2l+2} F\left(-2j, 2l+2; 4l+3; \frac{4}{2+mna_r}\right)$$

$$+\frac{m^2 k^2 \kappa_p}{(1-k^2)^2}\Gamma(2l+2)\left(\frac{na_r}{2+mkna_r}\right)^{2l+2} F\left(-2j, 2l+2; 4l+3; \frac{4}{2+mkna_r}\right)\Bigg].$$

Similarly, the fine structure potential with form factors gives rise to (15) where,

$$G_{FS}(n,l) = \left(\frac{2}{na_r}\right)^{2l+3}\frac{1}{2n2^{2(n-l-1)}}\sum_{j=0}^{n-l-1}\binom{2(n-l-j-1)}{n-l-j-1}\frac{\Gamma(4l+2j+3)}{j!\Gamma(2l+j+2)\Gamma(4l+3)}$$

$$\left[-\Gamma(2l)\left(1+\frac{\kappa_p}{(1-k^2)^2}\right)\left(\frac{na_r}{2+mna_r}\right)^{2l} F\left(-2j, 2l; 4l+3; \frac{4}{2+mna_r}\right)\right.$$

$$-\Gamma(2l+1)m\left(1+\frac{\kappa_p}{(1-k^2)^2}\right)\left(\frac{na_r}{2+mna_r}\right)^{2l+1} F\left(-2j, 2l+1; 4l+3; \frac{4}{2+mna_r}\right)$$

$$-\Gamma(2l+2)\frac{m^2}{2}\left(1+\frac{\kappa_p}{1-k^2}\right)\left(\frac{na_r}{2+mna_r}\right)^{2l+2} F\left(-2j, 2l+2; 4l+3; \frac{4}{2+mna_r}\right)$$

$$+\Gamma(2l)\frac{\kappa_p}{(1-k^2)^2}\left(\frac{na_r}{2+mkna_r}\right)^{2l} F\left(-2j, 2l; 4l+3; \frac{4}{2+mkna_r}\right)$$

$$\left.+\Gamma(2l+1)mk\frac{\kappa_p}{(1-k^2)^2}\left(\frac{na_r}{2+mkna_r}\right)^{2l+1} F\left(-2j, 2l+1; 4l+3; \frac{4}{2+mkna_r}\right)\right]. \quad (A.12)$$

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